Math 146C - Ordinary and Partial Differential Equations III - Spring 2011
April 28, 2011
Practice Midterm

Name: "Solutions"

Problem	Score
1	/25
2	/25
3	/25
4	/25
5	/25
6	/25
7	/25
8	/25
Score	/200

Problem 1 (25 points). *Find the Fourier series for the function*

$$f(x) = x^2, \quad -\pi \le x \le \pi. \qquad \Rightarrow L = \pi$$

f is even $\Rightarrow b_n = 0$

$$a_0 = \frac{2}{\pi}\int_0^\pi x^2\,dx = \frac{2}{\pi}\left(\frac{1}{3}x^3\Big|_0^\pi\right) = \frac{2\pi^2}{3}$$

$$a_n = \frac{2}{\pi}\int_0^\pi x^2 \cos nx\,dx \overset{\left(\substack{u=x^2 \\ dv=\cos nx\,dx}\right)}{=} \frac{2}{\pi}\left(\cancel{\frac{x^2 \sin nx}{n}\Big|_0^\pi} \xrightarrow{} 0 + \frac{1}{n}\int_0^\pi x \sin nx\,dx\right)$$

$$\overset{\left(\substack{u=x \\ dv=\sin nx\,dx}\right)}{=} \frac{2}{n\pi}\left[\frac{-x\cos nx}{n}\Big|_0^\pi - \frac{1}{n}\int_0^\pi \cos nx\,dx\right]$$

$$= \frac{2}{n\cancel{\pi}}\left(\frac{(-1)^{n+1}\cancel{\pi}}{n} - \cancel{\frac{1}{n^2}\sin nx\Big|_0^\pi} \xrightarrow{} 0\right) = \frac{2(-1)^{n+1}}{n^2}$$

$$\Rightarrow \boxed{f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n^2}\cos nx}$$

Problem 2 (25 points). *Find all the eigenvalues and eigenfunction for the boundary value problem*

$$x^2y'' - xy' + \lambda y = 0, \quad y(1) = y(\pi) = 0.$$

$$\lambda_n = 1 + \left(\frac{n\pi}{\ln(\pi)}\right)^2, \quad n \in \mathbb{N}$$

$$y_n = x\sin\left(\frac{n\pi \ln(x)}{\ln(\pi)}\right), \quad n \in \mathbb{N}$$

Problem 3 (25 points). *Find the solution of the inhomogeneous heat equation*

$$\begin{cases} u_{xx} = u_t + 2, \quad 0 < x < 1, \quad t > 0 \\ u(0,t) = 0, \quad u(1,t) = 0, \quad t > 0 \\ u(x,0) = x^2 - 2x + 2 \end{cases}$$

$$u(x,t) = x^2 - 2x + 1 + \sum_{n=1}^{\infty} \frac{\sqrt{2}\left(1-(-1)^n\right)}{n\pi} e^{-n^2\pi^2 t} \sqrt{2} \sin n\pi x$$

Problem 4 (25 points). *Find the series solution $u = u(r,\theta)$ of the Laplace equation*

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$$

inside the circle $r = a$ with boundary data $u(a,\theta) = f(\theta)$. Here f is a smooth function. Find an integral formula that equals the series solution.

$$u(r,\theta) = \frac{(a^2-r^2)}{2\pi}\int_0^{2\pi} \frac{h(\varphi)}{a^2 - 2ar\cos(\theta-\varphi) + r^2}\, d\varphi.$$

Problem 5 (25 points). *Let f be a square integrable, even function in $[-1,1]$ and $\sum_{n=1}^{\infty} b_n \cos n\pi x$ be its Fourier expansion. What is b_n? Prove Parseval's identity*

$$\sum_{n=1}^{\infty} b_n^2 = \int_{-1}^{1} f^2(x)\,dx$$

$$b_n = 2\int_0^1 f(x)\cos n\pi x\,dx = \int_{-1}^{1} f(x)\cos n\pi x\,dx$$

Since $f(x) = \sum_{n=1}^{\infty} b_n \cos n\pi x$

$$\Rightarrow f^2(x) = \sum_{n=1}^{\infty} b_n f(x)\cos n\pi x$$

$$\Rightarrow \int_{-1}^{1} f^2(x)\,dx = \sum_{n=1}^{\infty} b_n \int_{-1}^{1} f(x)\cos n\pi x\,dx = \sum_{n=1}^{\infty} b_n^2$$

Problem 6 (25 points). *Determine whether there is any value of the constant a for which the problem*

$$y'' + \pi^2 y = a - \cos \pi x, \quad y(0) = y(1) = 0,$$

has a solution. Find a series solution for each such value.

Values of a: $a = 0$

Solution: $y = c_2 \sin \pi x - \dfrac{x}{2\pi} \sin \pi x$

Problem 7 (25 points). *Consider the Sturm-Liouville problem*

$$[p(x)y']' - q(x)y + \lambda r(x)y = 0$$

on the interval $0 < x < 1$, *with the boundary value condition*

$$y(0) = y(1) = 0.$$

Here p, q, r *are smooth nonnegative functions. What does it mean that* λ *is an eigenvalue? Show all eigenvalues are real numbers.*

See theorem 11.2.1. $L[y] = -[p(x)y']' + g(x)y$

λ is an eval. for y if

$L[y] = \lambda r(x) y.$

Problem 8 (25 points).

(a) *Let* $f(x) = \begin{cases} x^{-\frac{1}{2}}, & 0 < x \le 1 \\ 0, & x = 0 \end{cases}$. *Show that* $\int_0^1 f(x)\,dx$ *exists as an improper integral, but* $\int_0^1 f^2(x)\,dx$ *does not.*

(b) *Let* $f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ -1, & x \in \mathbb{I}(= \mathbb{R} \setminus \mathbb{Q}) \end{cases}$. *Show that* $\int_0^1 f^2(x)\,dx$ *exists, but* $\int_0^1 f(x)\,dx$ *does not.*

(a) Easy. See math 9B.

(b) $f^2(x) = 1 \Rightarrow \int_0^1 f^2(x)\,dx = 1$

$$\overline{\int_0^1} f(x)dx = \lim_{n\to\infty} \sum_{m=1}^{n} \frac{1}{n} \max_{x\in\left[\frac{m-1}{n},\frac{m}{n}\right]}(f(x)) = \lim_{n\to\infty} \sum_{m=1}^{n} \frac{1}{n} = \lim_{n\to\infty} 1 = 1$$

$$\underline{\int_0^1} f(x)dx = \lim_{n\to\infty} \sum_{m=1}^{n} \frac{1}{n} \min_{x\in\left[\frac{m-1}{n},\frac{m}{n}\right]}(f(x)) = \lim_{n\to\infty} \sum_{m=1}^{n} \frac{-1}{n} = \lim_{n\to\infty} -1 = -1$$

$$\underline{\int_0^1} f(x)dx \neq \overline{\int_0^1} f(x)dx \Rightarrow \int_0^1 f(x)\,dx \text{ DNE}.$$